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Three-Dimensional Gas Dynamics in a Sphere

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In order to study gravitational collapse of a star leading to the formation of a super nova or a black hole and consequent emission of gravitational radiation, we have developed a numerical method to solve 3D partial differential equations in a sphere with various boundary conditions. We present the numerical method which is based on spectral analysis and application to the solutions of 3D wave equations and to the solutions of gas dynamics equations for pertinent cases. © 1990 Academic Press, Inc.

1. INTRODUCTION

In a previous paper [1], we described a numerical method for the solution of compressible hydrodynamics in spherical symmetry. Density and velocity were expanded in Chebychev polynomials series. The choice of spectral method was suggested by their ability to treat correctly any kind of bounary conditions, by their accuracy in the computation of spatial derivatives and by their ability to handle shock waves [1, 2].

Our aim being the study of gravitational wave during the collapse of a star, we had to build a code to solve 3D gas dynamic and wave equations. Let us recall that the physics of collapse can be 3D if the initial conditions have no symmetry or if a spontaneous symmetry breaking occurs. Moreover, it is well known that a spherical collapse does not give rise to emission of gravitational waves [3] and that the gravitational energy emitted by an axisymmetric collapse is too weak to be detected even by the next generation of gravitational waves detectors [4]. We therefore present a generalisation of our previous work to solve 3D gas dynamics and wave equations in spherical coordinates.

We have chosen to solve the equations in a "spherical-type" (spherical or ellipsoidal) coordinate system because of the regularity of the boundary surface associated with such a coordinates system and because of the simplicity of the matching of an internal solution with the corresponding external analytical one. Each quantity $Q(r, \theta, \varphi)$ is expanded in Fourier series for the longitudinal part, in Chebychev and/or Legendre polynomials series for the azimuthal part and in Chebychev polynomials series for the radial part. The method we present here allows us to handle rigorously the pseudo-singularities $r = 0$ and $\theta = 0, \pi$ typical of

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the "spherical-type" coordinates without artificial boundary conditions on these singular points.

In Section 2 we present in detail the basis of our method. We show how it is possible to remove the pseudo-singularities difficulty in taking into account the a priori known analytical properties of any tensorial physical quantities. Moreover, this technique leads to a numerical grid well adapted to the geometry and to an improvement of the accuracy of the solutions.

Application of this method to the resolution of wave equation for various boundary conditions is presented in Section 3. In Section 4, we describe an axisymetric and a 3D self-gravitating hydro code and give some results obtained in the solution of the gas dynamics equations.

2. The Method

In what follows, we assume that the basis of spectral methods in numerical computation is known. The reader not familiar with these techniques can find a description of their principles and their mathematical development in Gottlieb and Orszag [5] or in Canuto *et al.* [6], and an application to the study of spherical gravitational collapse in Ref. [1].

Let us consider a function $f(t, r, \theta, \varphi)$, with $r \in [0, r_b]$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi]$. Under very weak restrictions, f can be expanded in a Fourier series:

$$f(t, r, \theta, \varphi) = \sum_{m=0}^{\infty} a_m(t, r, \theta) e^{im\varphi}. \tag{1}$$

Now, each coefficient $a_m(t, r, \theta)$ is expanded in a series of independent functions $F_j(r)$, $G_l(\theta)$, namely,

$$a_m(t, r, \theta) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} a_{jlm}(t)\, F_j(r) G_l(\theta). \tag{2}$$

Usually, the family $G_l(\theta)$ is chosen to be the associated Legendre functions $P_l^m(\theta)$. Another choice for the set G_l could be Chebychev polynomials T_l which have some advantages (Fourier expansion cannot obviously be used because of the Gibbs phenomenon at the bounds of the interval due to the non-periodicity of the function).

Spherical harmonics

$$Y_l^m(\theta, \varphi) \stackrel{\text{def}}{=} P_l^m(\theta) e^{im\varphi} \tag{3}$$

are eigenvectors of the angular part of the Laplacian operator. This property allows easy inversion of this operator which appears in a lot of physical equations (particularly in the D'Alembertian operator and in the viscous terms of hydrodynamics equations). However, until now, from a numerical point of view, the Legendre transformation is performed by a product of a vector by a matrix which needs $\propto N^2$ operations, N being the number of the grid points, while the Chebychev transformation needs $\propto N \log N$ operations, since these transformations are executed by using a fast Fourier transformations algorithm. The efficiency of one of these methods with respect to the other depends on N and on the computer. We have chosen to expand the azimuthal part of the quantities in Chebychev polynomials of the first and second kinds. However, for the inversion of the Laplacian operator in 3D, we use an associated Legendre functions expansion in which the coefficients are computed from the Chebychev expansion coefficients by means of matrix multiplications. For the radial part, we have chosen the set $F_l(r)$ to be the Chebychev polynomials.

Consider now spectral expansions using Chebychev polynomials. A naive way to proceed is to seek an approximation of $f(t, r, \theta, \varphi)$, f_{JLM} say, of the form

$$f_{JLM}(t, r, \theta, \varphi) = \sum_{j=0}^{J} \sum_{l=0}^{M} \sum_{m=0}^{M} a_{jlm}(t) T_j(\psi_r) T_l(\psi_\theta) e^{im\varphi}, \tag{4}$$

where $T_j(\psi_r)$ and $T_l(\psi_\theta)$ are defined as

$$T_j(\psi_r) = \cos(j\psi_r), \qquad \text{with} \quad 1 - 2u = \cos \psi_r, \quad \psi_r \in [0, \pi]$$

and

$$T_l(\psi_\theta) = \cos(l\psi_\theta), \qquad \text{with} \quad 1 - \frac{2\theta}{\pi} = \cos \psi_\theta, \quad \psi_\theta \in [0, \pi], \tag{5}$$

where we have introduced the dimensionless variable u lying in the range $[0, 2]$.

The sampling grid associated with such an expansion is shown in Fig. 1a, ψ_r and ψ_θ being uniformly sampled. Let us recall that uniform sampling for the above variables must be used if FCT (fast Chebychev transforms) are employed. Moreover, the above grid corresponds to the zeros of the last Chebychev polynomial retained in the expansion and then it leads to an optimal accuracy of the computation of the coefficients a_{jlm} (see, e.g., [7]). The "pseudo-singularities" $r = 0$ and $\theta = 0$, π are treated by means of analytical continuation of the operators at these points. (See, for instance, an application of this method for the resolution of relativistic hydrodynamics around a Kerr Black Hole in Ref. [8].) As can be seen in Fig. 1a, these sampling points are concentrated near the polar axis and near the centre of the grid. It could be interesting to exploit this peculiarity in some particular problems (when high gradients of the solution are expected near this axis and/or near the centre of the grid), but, generally speaking, this accumulation of points is rather a disadvantage.

A way to overcome this difficulty is to take into account the known analytical properties of the solutions. Let us first remark that any function of the form

$$f(r, \theta, \varphi) = \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{l=0}^{L} \sum_{m=0}^{M} a_{jklm} r^j \cos^k \theta \sin^l \theta e^{im\varphi} \tag{6}$$

is not necessarily regular (consider for instance the simple case $f = \cos\varphi$, where f is a multi-valued function on the axis $\theta = 0$).

If f is a regular scalar function (i.e., in the context of the spectral approximation, $f \in \mathscr{C}^{\omega}$), f may be expanded as

$$f(r, \theta, \varphi) = \sum_i \sum_j \sum_k a_{ijk} \xi^i \bar{\xi}^j z^k, \tag{7}$$

where ξ, $\bar{\xi}$, z is a complex coordinate system related to the usual Cartesian one by

$$\xi = x + iy$$
$$\bar{\xi} = x - iy$$

with

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta. \end{aligned} \tag{8}$$

Taking account of the above relations, the expression (7) for f can be rewritten as

$$f(r, \theta, \varphi) = \sum_j \sum_k \sum_m a'_{jkm} r^{m+2j+k} \sin^{m+2j}\theta \cos^k \theta e^{im\varphi}, \tag{9}$$

where $m = i - j$.

It can be seen from (9) that if m is odd then

$$f_m(r, \theta) := \sum_j \sum_k a'_{jlm} r^{m+2j+k} \sin^{m+2j}\theta \cos^k\theta = f_m(r, \sin\theta) \tag{10}$$

and if m is even then

$$f_m(r, \theta) = f_m(r, \cos\theta). \tag{11}$$

Therefore we expand $f_m(r, \theta)$ in first kind Chebychev polynomials $T_l(\theta) = \cos(l\theta)$ for m even and in second kind Chebychev polynomials $T_l^{\dagger}(\theta) = \sin(l\theta)$ for m odd:

$$f_{2m+1}(r, \theta) = \sum_l b_{l,2m+1}(r) \sin(l\theta) \tag{12}$$

and

$$f_{2m}(r, \theta) = \sum_l b_{l,2m}(r) \cos(l\theta). \tag{13}$$

In a similar way, it can be noticed from (9) that the coefficients $b_{lm}(r)$ are even/odd polynomials for the even/odd values of l, respectively. The coefficients b_{lm} are then expanded as

$$b_{2l,m}(r)=\sum_i b_{2i,2l,m}T_{2i}(r) \tag{14}$$

and

$$b_{2l+1,m}(r)=\sum_i b_{2i+1,2l+1,m}T_{2i+1}(r), \tag{15}$$

the Chebychev polynomials $T_i(r)$ being those defined by Eq. (5), but, taking account of the known parity, the dimensionless variable u lies now in the range $[0, 1]$, corresponding to $\psi_r \in [\pi/2, \pi]$. The sampling points associated with this method of expansion is represented in Fig. 1b. Note that the above grid points are almost uniformly distributed along the radii near the centre of the box, while, for the naive sampling, the distance between two consecutive points near the centre is of order $1/N^2$, N being the number of points in the r-direction. The same conclusion holds for the θ-distribution. Finally, note that the coefficients $b_{l,m}$ behave as $r^l \sin^m\theta$. This regularity property can be used in particular problems where collocation is not necessary [9], however, in the general case, roundoff errors arising in computations when the number of degrees of freedom (in each direction) is greater than $\simeq 20$ forbid the use of this property. We have chosen to expand the quantities on Galerkin basis such that the coefficients $b_{l,m}$ behave as r^2 for l even and as r^3 for l odd. The same constraint is applied to the longitudinal part.

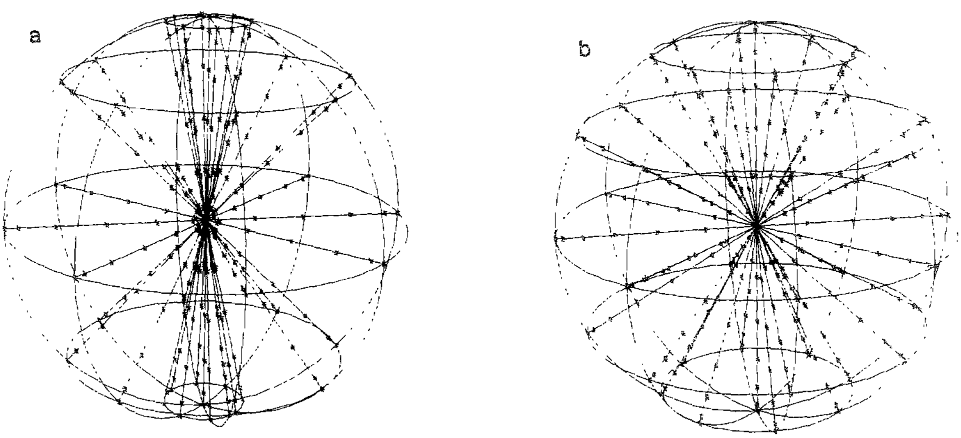


FIG. 1. (a) Naive sampling associated to the expansion defined by Eqs. (4) and (5). The numbers of degrees of freedom are 7 for r, 7 for θ, and 6 for φ. Note the condensation of points near the centre $r=0$ and near the axis $\theta=0, \pi$. (b) Smart sampling associated to the expansion defined by Eqs. (12)–(15). The numbers of points represented in this figure are the same as in Fig. 1a. This distribution of the grid points is more adapted to the geometry than in Fig. 1a. Note that the concentration of points in the r-direction near the boundary of the box improves the uniformity in the volume distribution.

3. Wave Equation

As an application of the numerical method described in the previous section let us consider the solution of the 3D scalar wav equation in flat space:

$$\frac{\partial^2 \Psi}{\partial t^2}=\frac{\partial^2 \Psi}{\partial r^2}+\frac{2}{r}\frac{\partial \Psi}{\partial r}+\frac{1}{r^2}\left(\frac{\partial^2 \Psi}{\partial \theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial \Psi}{\partial \theta}-\frac{1}{\sin^2\theta}\frac{\partial^2 \Psi}{\partial \varphi^2}\right) \tag{16}$$

for various initial and boundary conditions.

The calculations presented below were performed for the following reasons:

— having in mind to solve general relativistic stellar collapse, numerical solution of wave equation with outgoing-waves boundary conditions will give precious information on the stability of the temporal scheme.

— the D'Alembertian operator being a second-order operator both in space and in time directions for which analytical solutions are known, a numerical solution would provide a good test of the method.

— moreover, an expansion in Chebychev polynomials series requires that one treats the Laplacian operator implicitly both in the solution of the wave equation and in the treatment of the viscous part of the hydrodynamics equations in order to avoid excessively small time step (see [5]).

3.1. *Laplacian Inversion*

As has been explained in the previous section, quantities are generally expanded in Chebychev polynomials for the radial and the azimuthal parts. However, the fact that Legendre polynomials are eigenvectors of the Laplacian can be advantageously used in the case of the inversion of this operator. We have developed two kind of routines to invert the Laplacian operator. The first one is used to invert the Laplacian for quantities expanded in Chebychev polynomials in both the radial and the azimuthal part in 2D axisymetric problems. The second one is used to invert the Laplacian for full 3D quantities which are expanded in Chebychev polynomials for the radial part, in associated Legendre functions for the azymutal part and in Fourier series for the longitudinal part.

The capacity of the *super-computers* allows one to perform numerical 2D calculation using an average number of degrees of freedom, say 1024×1024 while a 3D calculation cannot be reasonably performed with more than $150 \times 40 \times 40$ points. It then follows that in the 2D case, computation of associated Legendre functions expansion is very time consuming (such a transformation needs $\propto N^2$ operations), while in the 3D case, this time becomes less important with respect to the time needed for the whole calculation.

The method we use to compute the associated Legendre functions expansion, to invert the Laplacian in practice and to treat boundary conditions, are given in detail in Appendix A.

3.2. *Temporal Scheme*

We first solved the 3D scalar wave equation in using the natural second-order temporal scheme,

$$\Psi^{j+1} = 2\Psi^{j} - \Psi^{j-1}$$
$$+ \frac{1}{2} dt^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right) \Psi^{j+1},$$
$$+ \frac{1}{2} dt^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right) \Psi^{j-1}, \quad (17)$$

where Ψ^j denotes the value of Ψ computed at the time t^j.

This scheme has been used with various initial and boundary conditions. We found that it is unconditionally stable for the boundary conditions $\Psi(1, \theta, \varphi) = C^{\text{ste}}$ (here and from now on r runs in the range $[0, 1]$). One of these calculations is presented in Fig. 2.

A most interesting application for our purpose is to solve the wave equation with outgoing wave boundary conditions:

$$\left.\frac{\partial\Psi}{\partial t} + \frac{\partial(r\Psi)}{\partial r}\right|_{r=1} = 0. \quad (18)$$

After second-order time discretization, the outgoing wave boundary conditions can be expressed as

$$3\Psi^{j+1} + 2\,dt\,\frac{\partial\Psi^{j+1}}{\partial r} = 4\Psi^{j} - \Psi^{j-1}. \quad (19)$$

We discovered that the previous numerical scheme is unconditionally stable for the even l, where l is the azimuthal quantum number (Eqs. (12)–(13)), while it becomes unstable for the odd l when the time step becomes too small for a given number of degrees of freedom or, conversely, when the number of degrees of freedom is too small for a given dt. (The same kind of instability arises also for the boundary condition $\partial\Psi/\partial r|_{r=1} = 0$.) Note that this instability does not arise in calculations with equatorial symmetry.

In order to understand this strange behaviour, which appears either with Chebychev and associated Legendre functions expansion for θ, we considered the system,

$$\frac{\partial\Psi}{\partial t} = -U + \nu\,\Delta\Psi$$
$$\frac{\partial U}{\partial t} = -\Delta\Psi. \quad (20)$$

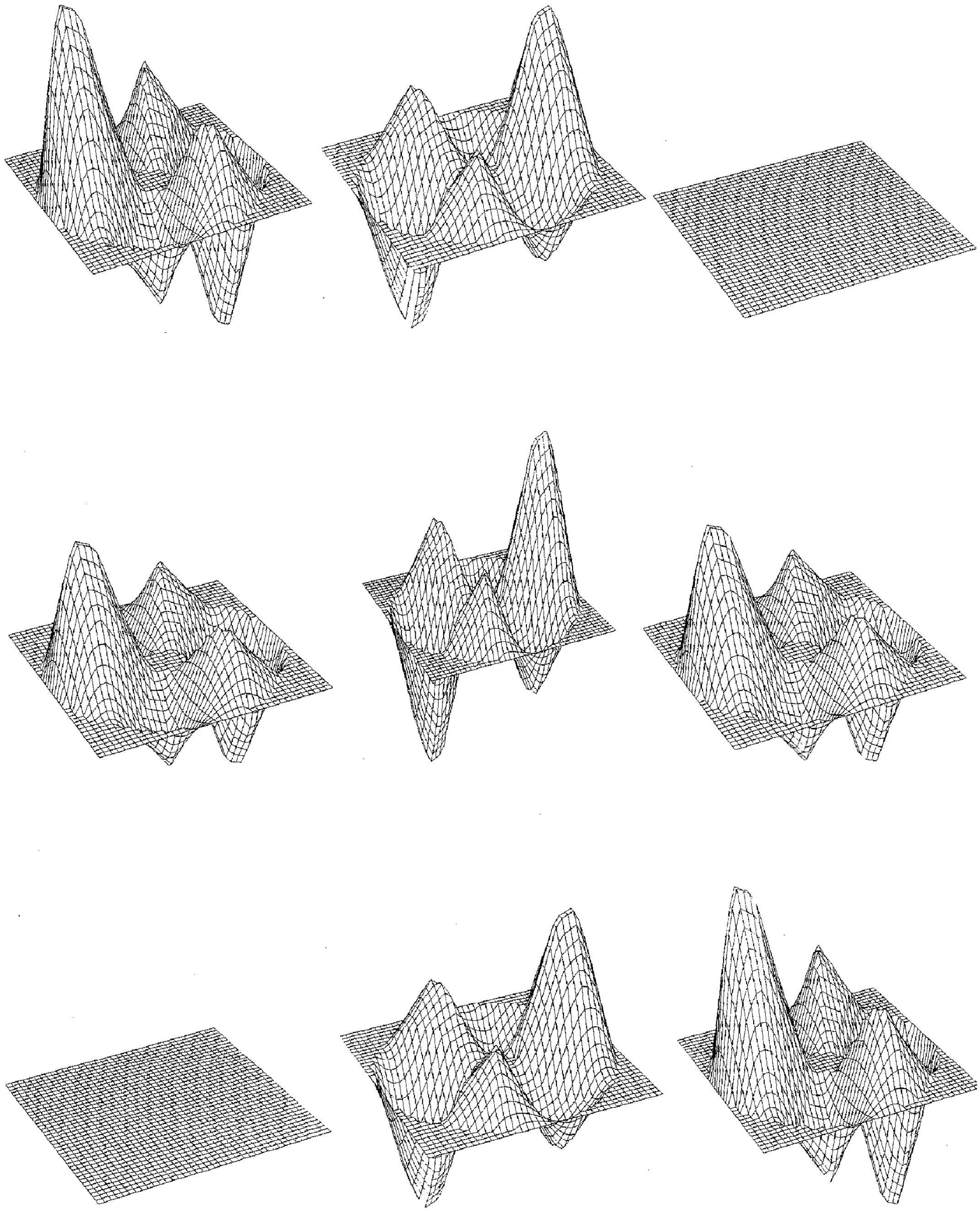


FIG. 2. Time evolution of a 3D stationary wave. The initial condition is an eigenfunction of the Laplacian. The figures represent the $m = 0$ mode evolution of the wave during one period.

It is to be noted that in the case $\nu = 0$ this system is equivalent to the wave equation and, moreover, that this system corresponds to the linearised compressible fluid dynamics equations in which the velocity $\mathbf{v}$ satisfies $\mathbf{v} = \text{grad}\ \Psi$.

A second-order temporal scheme can be

$$\begin{aligned} \Psi^{j+1} &= \Psi^j - \frac{dt}{2}(U^{j+1} + U^j) + \nu\frac{dt}{2}(\Delta\Psi^{j+1} + \Delta\Psi^j) \\ U^{j+1} &= U^j + \frac{dt}{2}(\Delta\Psi^{j+1} + \Delta\Psi^j). \end{aligned} \tag{21}$$

After elimination of U^{j+1} in the first equation, we obtain

$$\begin{aligned} \Psi^{j+1} &= \Psi^j - dt\, U^j + \frac{dt}{2}\left(\nu + \frac{dt}{2}\right)(\Delta\Psi^{j+1} + \Delta\Psi^j) \\ U^{j+1} &= U^j - \frac{dt}{2}(\Delta\Psi^{j+1} + \Delta\Psi^j). \end{aligned} \tag{22}$$

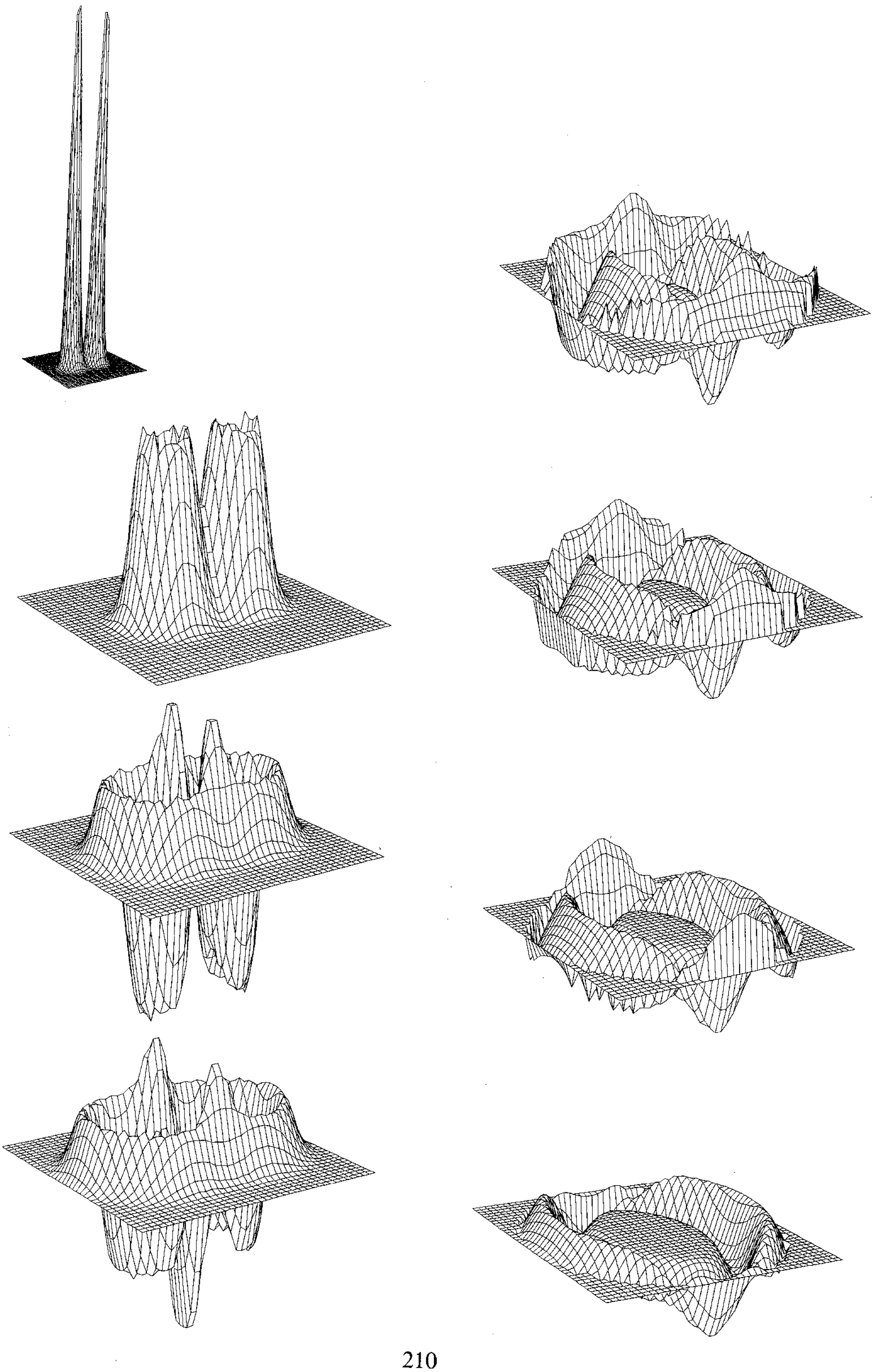
The first equation is solved implicitly for Ψ, while the second equation gives the value of U at the time t^{j+1} by means of the computed value of Ψ at the time t^{j+1}. The boundary conditions are imposed in the first equation. We found the *same kind of instability* for $\nu = 0$.

For practical purposes, this instability can be removed with a not too large coefficient of viscosity ν. Finally, bearing in mind that the outgoing wave boundary conditions as expressed in Eq. (18) is an asymptotic one, small reflections must arise at the boundary of the numerical grid. Such a small viscosity ν damps the reflected part of the wave.

In order to give an explanation of this instability, let us describe the phenomenon. Consider a run for which the initial conditions are such that the energy of the wave is concentrated near the centre of the box. Observation of the time evolution of the wave shows that, when the wave reaches the boundary, a boundary layer seems to arise for the antisymetric part (i.e., for the odd values of l). The thickness of this layer depends on the value of the coefficient $(\nu + dt/2)$ which appears in front of the Laplacian in Eq. (22).

If the number of degrees of freedom is not large enough to describe the layer, an instability arises. However, at the present time, we do not understand why this instability appears only for the odd values of l. Note that the number of degrees of freedom for the r-expansion is odd for even values of l and even for odd values of l. Maybe this is the reason for the above behaviour.

A run for the propagation of waves with outgoing wave boundary conditions is presented in Fig. 3.



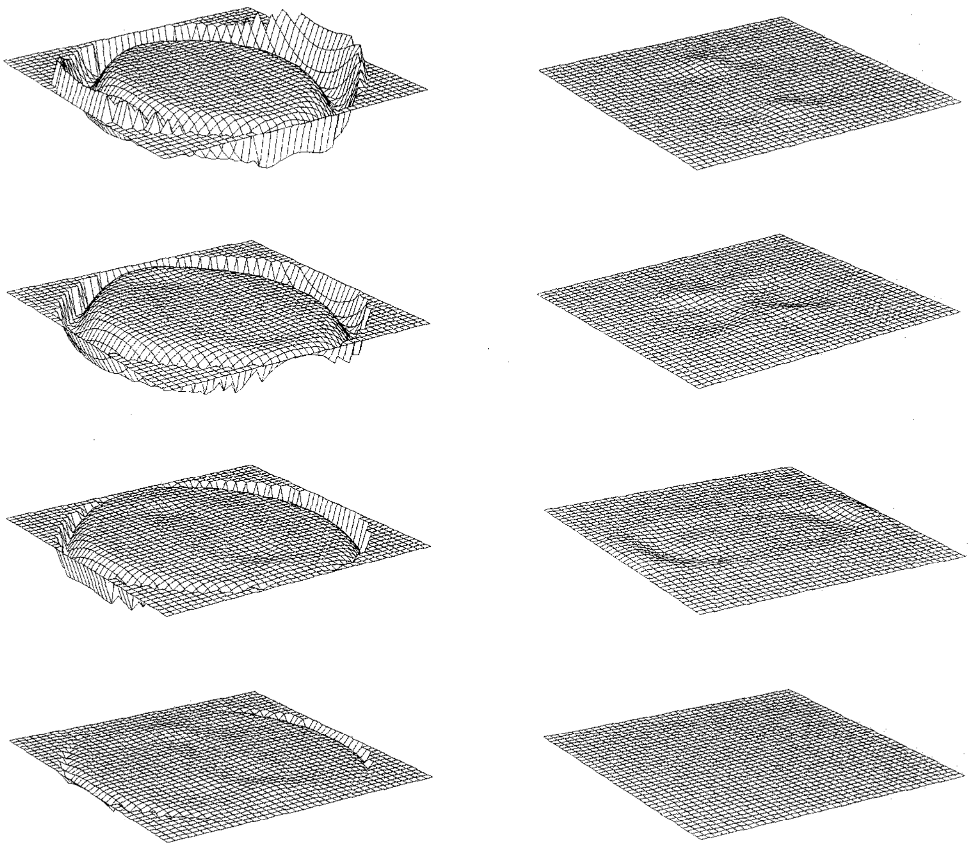


FIG. 3. Time evolution of a 3D scalar wave with outgoing boundary conditions during two crossing times. These figures represent $\Psi(t, r, \theta, \varphi)\,|_{y=0}$. Note a small reflection of the wave near the boundary. This reflection is not a numerical one, but is due to the fact that the outgoing boundary condition is asymptotic.

4. HYDRODYNAMICS EQUATIONS

We recall that, for the reasons already explained in Section 1, we are obliged to use a numerical grid such that its boundary is a smooth, regular, and convex surface having the topology of the sphere. In this case, it is always possible to reduce this surface to a sphere by means of an appropriate coordinate transformation. We will then consider in what follows a spherical numerical box, the physical quantities being described with respect to the usual spherical coordinate system (r, θ, φ).

The coordinate system being now chosen, it remains to choose the way to describe the physical quantities. It is to be noticed that a scalar quantity is everywhere a single-valued function, while the components of a tensor can be multidefined functions on some part of the space if tensors are decomposed on a singular tensorial basis. This remark holds for the case of the components of a vector field with respect to the natural triad associated to the spherical coordinates system.

Consider for instance a vector field $\mathbf{v}$. Writing $\mathbf{v}$ as

$$\begin{aligned}\mathbf{v} &= v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \\ &= v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\varphi \mathbf{e}_\varphi,\end{aligned} \tag{23}$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ is the canonical orthonormal triad associated to the spherical coordinates, it can be easily seen that the θ- and the φ-components of $\mathbf{v}$ are multivalued function of (r, θ, φ) on the axis $\theta = 0, \pi$. However, if the vector field $\mathbf{v}$ is regular, these components have to satisfy on the axis the regularity conditions

$$\begin{aligned}v_\theta &= \cos \varphi v_x + \sin \varphi v_y \\ v_\varphi &= -\sin \varphi v_x + \cos \varphi v_y.\end{aligned} \tag{24}$$

Note that this trouble arises worthly at the origin $r = 0$.

Even if it is possible to treat this kind of singularity in taking account of the above regularity conditions (see, e.g., Ref. [8]), a better way to overcome this difficulty is to express the tensorial physical quantities by means of their components with respect to a regular triad (for instance, the cartesian one) considered as scalar functions of r, θ, and φ. The choice of the triad must be determined in function of each particular case as it will be illustrated below. The same choice has been made for the same reasons by the Japanese group [10].

4.1. 3*D Code*

4.1.1. *General Considerations*

The quantities used in this code are the mass density ρ, the three cartesian components v_x, v_y, v_z of the velocity $\mathbf{v}$ and the gravitational potential Φ expressed with respect to the spherical coordinate system.

The equations to be solved are the momentum conservation

$$\frac{\partial v_i}{\partial t} = -v_x \frac{\partial v_i}{\partial x} - v_y \frac{\partial v_i}{\partial y} - v_z \frac{\partial v_i}{\partial z} - \frac{1}{\rho} \partial_i P - \partial_i \Phi + \nu \Delta v_i, \tag{25}$$

where $i = x, y, z$, the mass conservation equation,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z}, \tag{26}$$

and the Poisson equation,

$$\Delta\Phi = -4\pi G\rho, \tag{27}$$

where $P = P(\rho)$ is the pressure and G is the gravitational constant. The method to solve the Poisson equation is described in Appendix.

The dissipative terms are written in the form $\nu\,\Delta v_i$. We want to emphasize that these are not the correct dissipative term since the entropy production terms are not positive definite. The correct contribution would be $(\xi/\rho)(\Delta v_i + (1/3)\partial_i \operatorname{div} \mathbf{v})$, ξ being the dynamical viscosity which is supposed to be independent of ρ.

We use the first form of the dissipative terms for simplicity and because we are not interested at the present time in the entropy production. These terms are used only in order to smooth shocks. At the present time, the code uses a constant (in space and time) numerical value of ν, which is high enough to spread out the shock on about three grid points. For physical applications, we shall use a viscous coefficient function of space and time analog to the so-called artificial viscosity used by most of the finite differences codes. However, in a realistic physical situation, the correct dissipative terms would have to be considered. This introduces a complication due to the coupling of Navier–Stokes equations via the bulk viscosity terms. This difficulty may be overcome by simultaneously solving all velocity components. Such a scheme has been coded for the case of 3D gas dynamics with periodic boundary conditions. It was found that the code is stable for any value of ξ.

Note that, for this problem, spectral methods need viscosity. This is not a weakness of the method, but rather corresponds to a physical reality. Methods that do not need viscous terms can work only because an undercontrolled hidden intrinsic viscosity is present.

It is to be noticed that the equations written in this form hide the trouble of spherical coordinate singularities. For instance, the simple operator $\partial/\partial x$ reads

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\,\frac{\partial}{\partial r} + \cos\theta\cos\varphi\,\frac{1}{r}\frac{\partial}{\partial\theta} - \sin\varphi\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{28}$$

Even though $\partial/\partial x$ is regular, some operators of the r.h.s. of the previous relation are not. The singularities appear in the evaluation of $(1/r)(\partial/\partial\theta)$ and of $(1/r\sin\theta)(\partial/\partial\varphi)$. However if Q is a $\mathscr{C}^\omega$ class function, Q has to satisfy the regularity conditions

$$\frac{\partial Q}{\partial\theta} = r\left(\cos\theta\cos\varphi\,\frac{\partial Q}{\partial x} + \cos\theta\sin\varphi\,\frac{\partial Q}{\partial y} - \sin\theta\,\frac{\partial Q}{\partial z}\right) \tag{29}$$

and

$$\frac{\partial Q}{\partial\varphi} = (r\sin\theta)\left(-\sin\varphi\,\frac{\partial Q}{\partial x} + \cos\varphi\,\frac{\partial Q}{\partial y}\right) \tag{30}$$

which ensure that numerical evaluation of the previous terms gives a finite value. Note that our formalism in expanding scalar quantities takes into account these regularity properties. Another critical point arises from the fact that each one of the three operators which appear in $\partial/\partial x$ applied to a scalar function Q is no more a function in the sense that it is multi-valued on the axis $\theta=0, \pi$. Nevertheless the sum of these three terms is a regular scalar quantity. Consider the simple [illegible] $Q_\theta=\cos^2\varphi\cos^2\theta$, and $Q_\varphi=\sin^2\varphi$. It is obvious that, on the axis, Q_θ and Q_φ are multi-defined, but that $Q_r+Q_\theta+Q_\varphi$ is regular.

Consequently, from a numerical point of view, all these three terms cannot be computed independently.

4.1.2. *Finite Difference Temporal Scheme*

We use a second-order scheme in which the source and the advective terms are treated explicitly and the dissipative terms implicitly. Each of the quantities that appear in the r.h.s. of the equations are computed at the time $t^{j+1/2}$ except for the Laplacian terms that are treated implicitly. More precisely, div $\rho\mathbf{v}$, which appears in the mass conservation equation, and the quadratic terms $\mathbf{v}\cdot\operatorname{grad}\mathbf{v}$, are computed by extrapolation of their values at the times t^j and t^{j-1}. Conversely, the force terms $(1/\rho)\partial_i P$ and $\partial_i\Phi$ are computed by interpolation from their values at the times t^j and t^{j+1}. This means that the conservation equation has to be solved before the other ones.

The temporal scheme then reads:

$$\rho^{j+1}=\rho^j-dt\left(\frac{3}{2}\operatorname{div}\rho\mathbf{v}^j-\frac{1}{2}\operatorname{div}\rho\mathbf{v}^{j-1}\right), \tag{31}$$

$$\rho^{j+1/2}=\frac{1}{2}(\rho^{j+1}+\rho^j), \tag{32}$$

$$\Phi^{j+1/2}=-4G\Delta^{-1}\rho^{j+1/2}, \tag{33}$$

$$P^{j+1/2}=P(\rho^{j+1/2}), \tag{34}$$

$$v_i^{j+1}=v_i^j+dt\times\left(-\frac{3}{2}\mathbf{v}\cdot\operatorname{grad}v_i^j+\frac{1}{2}\mathbf{v}\cdot\operatorname{grad}v_i^{j-1}\right.$$
$$\left.-\frac{1}{\rho^{j+1/2}}\partial_i P^{j+1/2}-\partial_i\Phi^{j+1/2}+\frac{1}{2}\nu(\Delta v_i^{j+1}+\Delta v_i^j)\right). \tag{35}$$

If the dissipative terms are writen in an exact way, the modifications introduced in the momentum conservation equation impose a semi-implicit treatment of the Laplacian. The previous numerical scheme has then to be modified as

$$v_i^{j+1} = v_i^j + dt\ \text{Source} + dt\ \xi \left(\frac{1}{\rho} - \frac{1}{\rho_{\min}}\left(\Delta v_i + \frac{1}{3}\partial_i \partial_n v_n\right)\right)^{j+1/2}$$

$$+ dt \frac{\xi}{\rho_{\min}}\left(\frac{2}{3}(\Delta v_i^{j+1} + \Delta v_i^j) - \frac{1}{3}\sum_{n\neq i} \partial_n^2 v_i + \partial_i \sum_{n \neq i} \partial_n v_n^{j+1/2}\right). \qquad (36)$$

The boundary conditions are imposed on the components of the velocity v_x, v_y, v_z by using the τ-approximation [11]. The code is able to treat any boundary condition of the form

$$\alpha(t)\frac{\partial v_i}{\partial r} + \beta(t) v_i = \gamma(t, \theta, \varphi), \qquad (37)$$

where $i = x, y, z$. Note that it is not possible to treat in a direct way a boundary condition of the type

$$\alpha(t, \theta, \varphi)\frac{\partial v_i}{\partial r} + \beta(t, \theta, \varphi)\, v_i = \gamma(t, \theta, \varphi). \qquad (38)$$

However, with a small complication, it is possible to tackle the kind of boundary conditions written in equation (38) applied to the spherical components v_r, v_θ, v_φ of the velocity. Note that, for our present purpose, such a complication can be avoided.

4.2. "2.5*D*" *Code*

4.2.1. *General Considerations*

We call "2.5D" code a code able to handle axisymetric configuration with rotation around the axis of symmetry. Of course, this is a particular case that could be treated by the 3D code, but, it will then result in a waste of computational time because this code cannot obviously take into account the symmetries of the problem. Recall that, even some quantities (e.g., ρ, $\|\mathbf{v}\|$, ...) do not depend on φ, other unknown functions (e.g., v_x, v_y, v_z) depend on r, θ, and φ.

For axisymetric calculations, we will then keep the spherical coordinate system to describe the scalar quantities, but, to expand the vector fields, we choose another triad of vectors which takes into account the symmetries. We have chosen the canonical orthonormal triad associated with the cylindrical coordinates system. Writing v_ρ, v_φ, v_z the components of the velocity with respect to this triad, we have

$$\begin{aligned} v_\rho &= \cos\varphi v_x + \sin\varphi v_y \\ v_\varphi &= \sin\varphi v_x - \cos\varphi v_y \\ v_z &= v_z. \end{aligned} \qquad (39)$$

Introducing the operators ∇_ρ, ∇_z, and Δ_m,

$$\nabla_\rho = \sin\theta \frac{\partial}{\partial r} + \cos\theta \frac{1}{r}\frac{\partial}{\partial\theta}, \tag{40}$$

$$\nabla_z = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{41}$$

and

$$\Delta_m = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{m^2}{\sin^2\theta}\right), \tag{42}$$

the hydrodynamics equations read

$$\frac{\partial v_\rho}{\partial t} = -(v_\rho \nabla_\rho v_\rho + v_z \nabla_z v_\rho) - \frac{v_\varphi^2}{r\sin\theta} - \frac{1}{\rho}\nabla_\rho P - \nabla_\rho \Phi + \nu\Delta_1 v_\rho, \tag{43}$$

$$\frac{\partial v_\varphi}{\partial t} = -(v_\rho \nabla_\rho v_\varphi + v_z \nabla_z v_\varphi) - \frac{v_\rho v_\varphi}{r\sin\theta} + \nu\Delta_1 v_\varphi, \tag{44}$$

$$\frac{\partial v_z}{\partial t} = -(v_\rho \nabla_\rho v_z + v_z \nabla_z v_z) - \frac{1}{\rho}\nabla_z P + \nu\Delta_0 v_z, \tag{45}$$

$$\frac{\partial \rho}{\partial t} = -\left(\nabla_\rho(\rho v_\rho) + \frac{\rho v_\rho}{r\sin\theta} + \nabla_z(\rho v_z)\right), \tag{46}$$

$$\Phi = -4\pi G \Delta_0^{-1}\rho, \tag{47}$$

$$P = P(\rho). \tag{48}$$

It is to be noticed that the components v_ρ and v_φ are the component of $\mathbf{v}$ with respect to a singular triad. It then follows that these components are multi-defined functions and, as can be seen from relations (39), that these quantities must be expanded in the θ-direction as $\sum_l a_l(t, r)\sin l\theta$. Moreover, the evolution equation for v_φ is linear. It is then possible to solve it without the need of viscosity

4.2.2. *Finite Difference Temporal Scheme*

In order to ensure that the numerical scheme used in the 3D code is stable, the time-step has to satisfy the condition

$$dt < \mathrm{Min}\left(\frac{1}{|v_b|\, N^2}, \frac{1}{|v_{\max}|\, N}\right), \tag{49}$$

where N is the number of degrees of freedom in the r-direction, v_b is the radial component of the velocity at the boundary, and $v_{\max}$ is its maximum value [5]. Note that the relation (49) is verified for large values of N, but for small values ($N \leqslant 65$)

and for the particular case in which we are interested, the stability criterion is satisfied for

$$dt \propto \mathrm{Min}\left(\frac{1}{|v_b|\, N^{1.2}}, \frac{1}{|v_{\max}|\, N}\right), \tag{50}$$

(see Fig. 4). It is then obvious that, when the number of points becomes large, as can be the case for the 2.5D code, this condition becomes prohibitive. The way to overcome this unpleasant fact is to treat the operators semi-implicitly.

Let us consider each term of the r.h.s. of Eq. (42). The terms that may cause trouble are those in which $\partial/\partial r$ appear because of the concentration of points near the boundary and the terms in which $(1/r)(\partial/\partial\theta)$ appears since, near the centre of the box, these operators behave as $\partial^2/\partial r\partial\theta$.

The first class of terms mentioned above must be treated in a semi-implicit way because of the Chebychev expansion, while the same kind of treatment must be applied to the second one too because of the use of the spherical coordinate system. Below we review all these terms that appear in the evolution equation of v_ρ and give the corresponing temporal scheme we use.

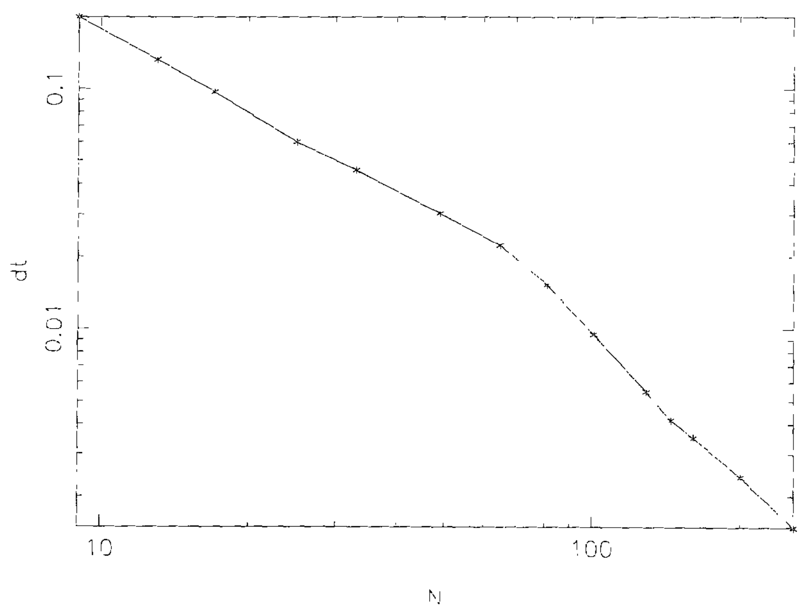
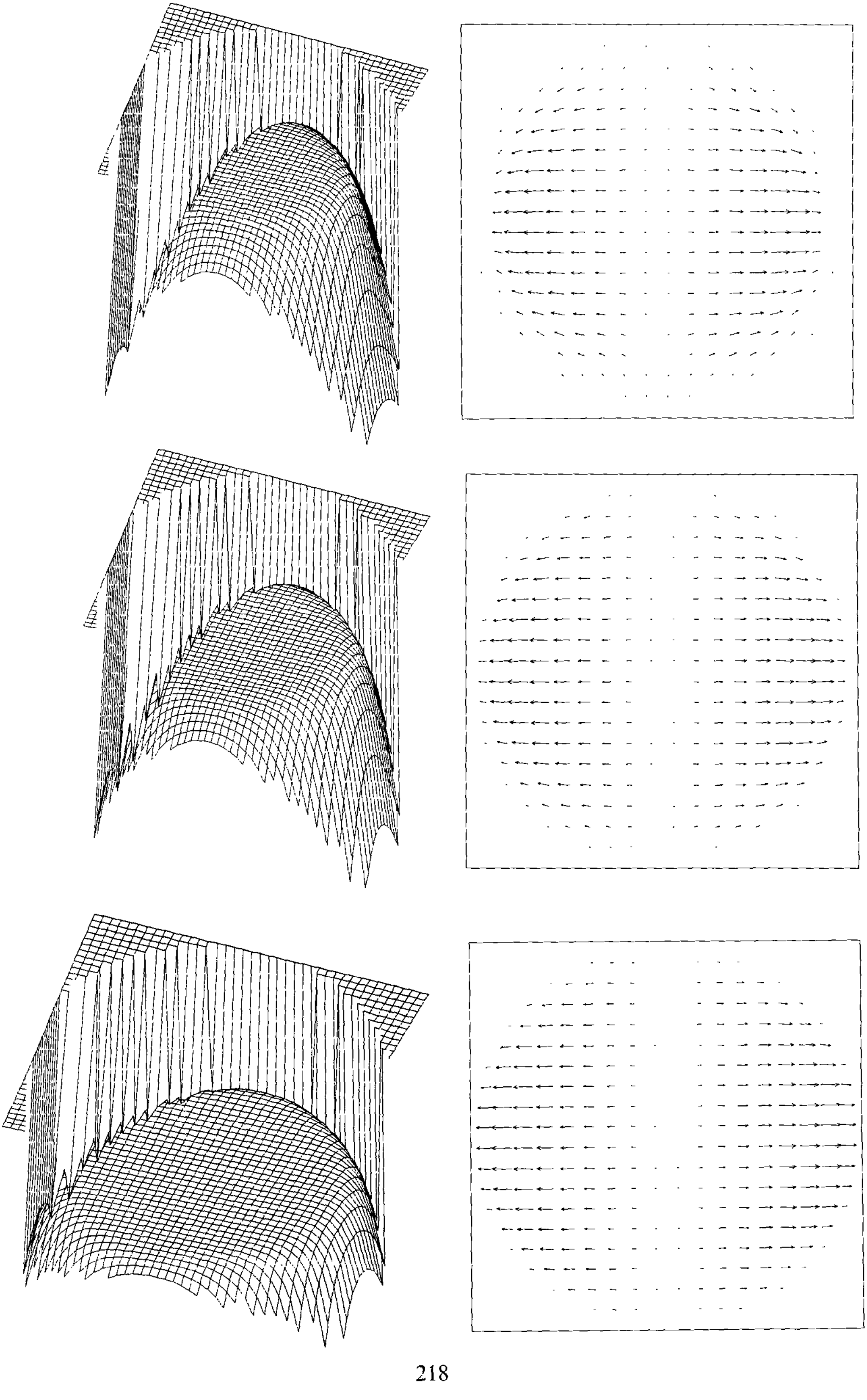


FIG. 4. Plot in log–log scale of the maximum time-step versus the number of degrees of freedom allowed to ensure numerical stability for an explicit temporal scheme for the resolution of $\partial f/\partial t = -\sin x(\pi/2)(\partial f/\partial x)$; $x \in [0, 2]$. This curve shows that $dt_{\max} \propto 1/N^{1.2}$ for $N < 65$ and $dt_{\max} \propto 1/N^2$ for $N > 65$.



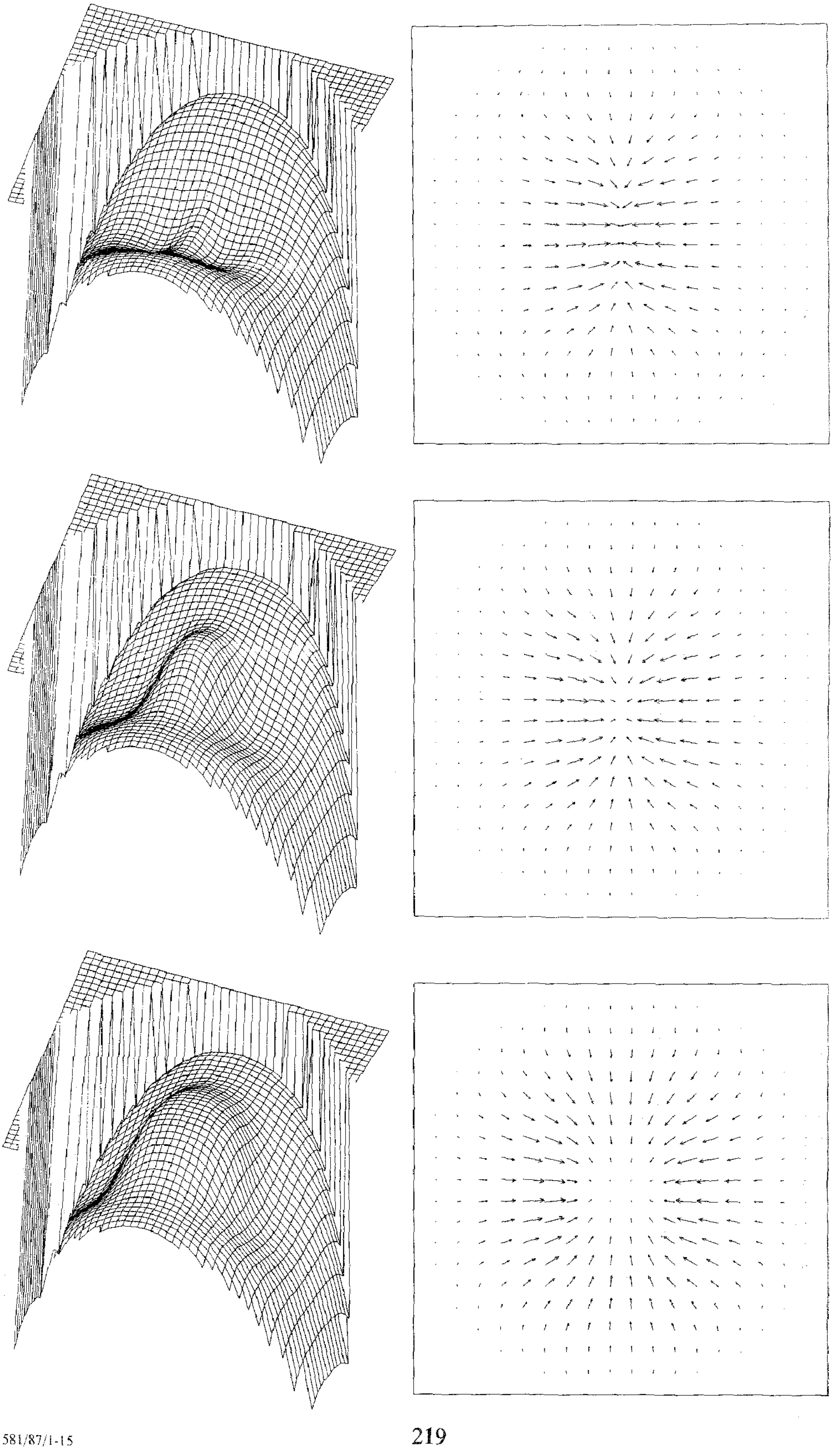
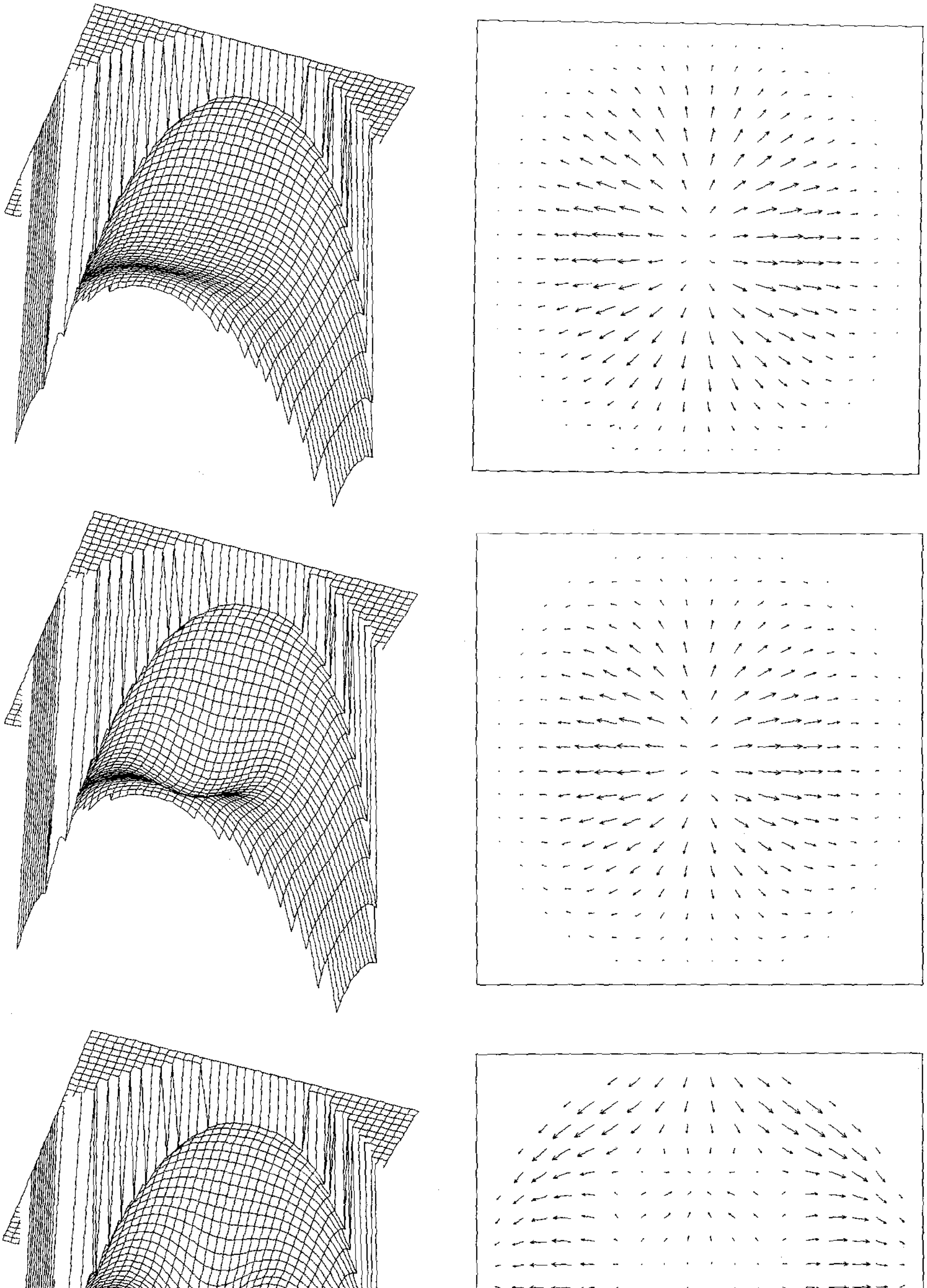


FIGURE 5—*Continued*



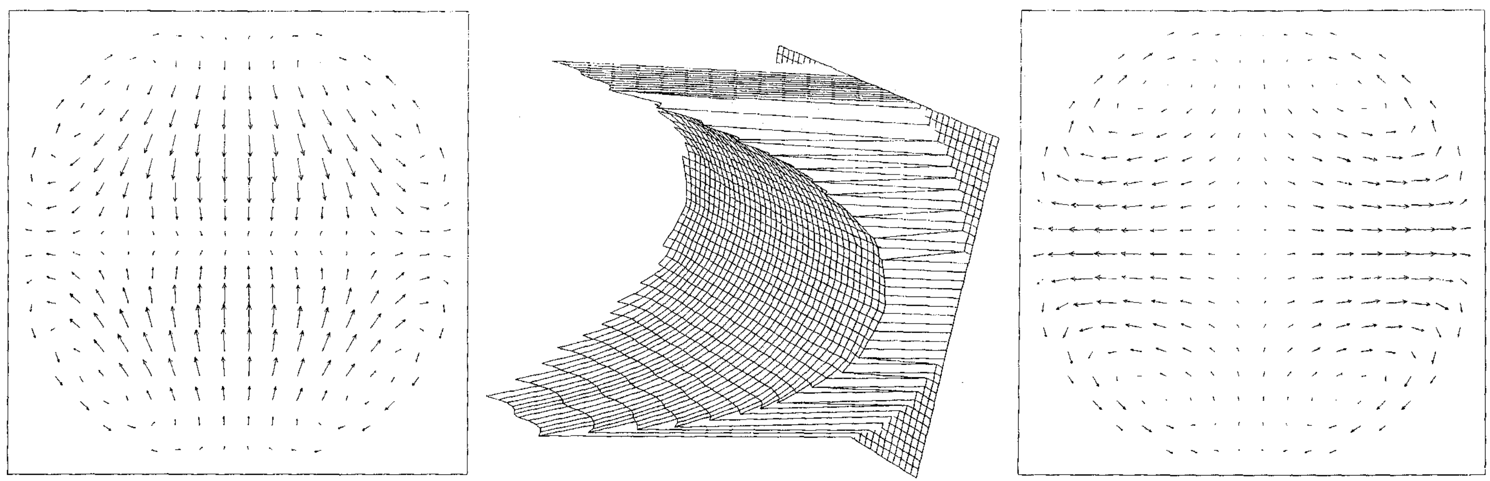


FIG. 5. Time evolution of density and velocity field of a 2.5D axisymmetric compressible hydrodynamics calculation with initial conditions $\rho = C^{\text{ste}}$, $v_\rho = v_z = 0$, and $v_\varphi = \omega r$. The parameters of this simulation, expressed in dimensionless units, are the adiabatic index $\gamma = 5/3$, the sound velocity $c_s = \sqrt{5/3}$, the acoustic crossing time $\tau_c = D/c_s = 1.6$, where D is the diameter of the box, the rotational time at $t = 0$, $\tau_r = 1.8$, and the diffusion time $\tau_D = D^2/\nu = 40$.

The semi-implicit scheme is

$$
\begin{aligned}
v_\rho^{j+1} = v_\rho^j + dt \\
&\times\Bigg(-\left[(v_\rho \sin\theta + v_z\cos\theta - \alpha_0(t,\theta) - \alpha_1(t,\theta)r - \alpha_2(t,\theta)r^2)\frac{\partial v_\rho}{\partial r}\right]^{j+1/2} \\
&-\frac{1}{2}\left[\alpha_0(t,\theta) + \alpha_1(t,\theta)r + \alpha_2(t,\theta)r^2)\frac{\partial v_\rho}{\partial r}\right]^{j+1} \\
&-\left[v_\rho\frac{\cos\theta}{r}\frac{\partial v_\rho}{\partial r}\right]^{j+1/2} + \left[\frac{v_\varphi^2}{r\sin\theta}\right]^{j+1/2} \\
&+\left[(v_z - \beta(t))\frac{\sin\theta}{r}\frac{\partial v_\rho}{\partial\theta}\right]^{j+1/2} \\
&+\frac{1}{2}\left[\beta(t)\frac{\sin\theta}{r}\frac{\partial v_\rho}{\partial\theta}\right]^{j+1} + \frac{1}{2}\left[\beta(t)\frac{\sin\theta}{r}\frac{\partial v_\rho}{\partial\theta}\right]^{j} \\
&-\frac{1}{2}\left(\frac{\sin\theta}{\rho}\frac{\partial}{\partial r} + \frac{\cos\theta}{r\rho}\frac{\partial}{\partial\theta}\right)(P^{j+1} + P^j) + \frac{\nu}{2}\Delta_1(v_\rho^{j+1} + v_\rho^j)\Bigg),
\end{aligned}
\tag{51}
$$

where each quantity at the time $j+1$ is computed implicitly and each quantity written at the time $j+1/2$ is computed explicitly either by extrapolation from its value at the times j and $j-1$ or by interpolation from its values at the times j and $j+1$. The same kind of treatment holds for the v_φ, v_z, and ρ equations of evolution.

The functions $\alpha_i(t,\theta)$ are chosen in such a way that the advective terms vanish at the boundary $r=1$ following the regularity rules. The function $\beta(t)$ is chosen in such a way that $\beta = v_z|_{r=0}$ (note, that, according to the regularity rules, v_z does not depend on θ at $r=0$. Recalling that v_φ and v_ρ vanish at $r=0$, it follows that $v_\varphi^2/r\sin\theta$ and $(v_\rho\cos\theta/r)(\partial v_r/\partial\theta)$ are not dangerous).

The operator $\Delta_1 - \beta(\sin\theta/r)(\partial/\partial\theta)$ must be inverted in the expansion coefficients space for both r and θ. On the other hand, the operator $(\alpha_0(\theta) + r\alpha_1(\theta) + r^2\alpha_2(\theta))(\partial/\partial r)$ cannot be, because of the dependence of the α_i on θ. This operator must then be inverted in the coefficient space for r and in the physical space for θ. To overcome this difficulty, the integration in time can be performed usng a generalisation of ADM (alternating direction method, see, e.g., Ref. [12]). The pressure terms at the time $j+1$ are treated in the same way as shown in Section 3 for the wave equation (Eqs. (21)–(22)).

4.3. *Numerical Results*

A 2.5D hydrodynamics calculation, without gravitational field, is presented in Figs. 5. The initial conditions are

$$
\begin{aligned}
\rho(t_0, r, \theta) &= C^{\mathrm{ste}}, \\
v_\rho(t_0, r, \theta) &= v_z(t_0, r, \theta) = 0, \\
v_\varphi(t_0, r, \theta) &= \omega r,
\end{aligned}
\tag{52}
$$

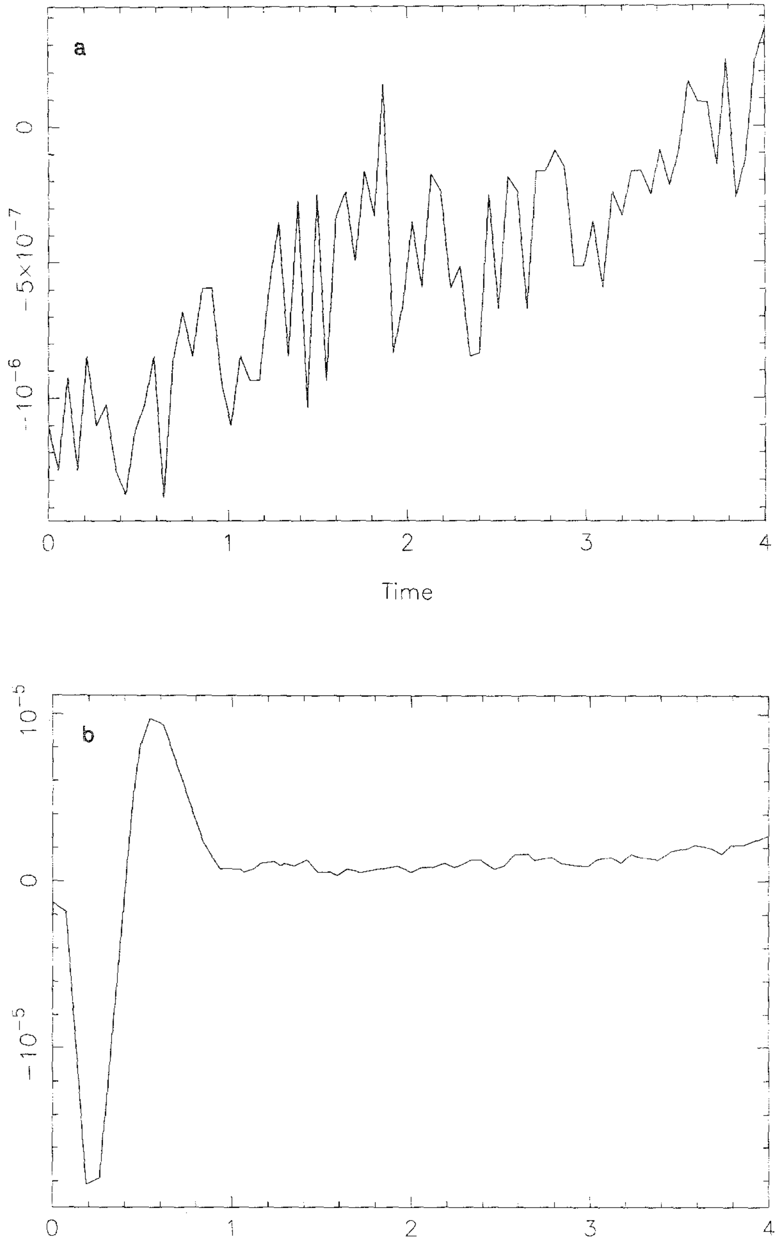


FIG. 6. Relative error of mass conservation (*a*) and *z*-components of the angular momentum (*b*) versus time, during the simulation presented in Fig. 5.

with the boundary conditions

$$v_\rho(t, 1, \theta) = v_z(t, 1, \theta) = 0. \tag{53}$$

In this case, the viscous coeffcient which appears in the v_φ-equation of evolution was set to zero.

It can be seen that the matter first tends to go outside the box and is reflected on the boundary of the box. After this first reflection, a soft shock begins to form (Fig. 5). The evolution of the mass and of the z-component of the angular momentum (which in this case have to be conserved) are represented in function of time in Fig. 5b and c. This calculations used a 33×33 numerical grid points and was executed on the VAX-8600 of the Observatoire de Meudon. The CPU time per time step is 2.56 s.

In order to test our codes, we performed the same run with the 3D code. This run gave the same result within the roundoff errors, namely, 10^{-6} for a calculation in single precision.

5. Conclusion

We have described in this paper a numerical method for solving 3D and 2D systems of partial differential equations in a "spherical-type" coordinate system by means of a spectral expansion. Our method takes into account the analytical properties of the physical quantities.

Applications to the resolution of 3D wave equation and Newtonian hydrodynamics are shown. Each quantity is expanded in Fourier series for the longitudinal part in Chebychev polynomials of the first or second kind or associated Legendre functions for the azimuthal part and in even or odd Chebychev polynomials for the radial part, the associated Legendre function expansion being used only in the 3D case to invert the Laplacian operator.

Moreover, we have shown that, with an appropriate decomposition of the vector fields, it is possible to avoid the difficulties attached to the spherical components of the vector fields.

The actual capacity of the *super-computers* does not allow to use more than a hundred points in each direction for the 3D case while it allows a thousand points in each direction for the 2D case. On the other hand, stability of an explicit temporal scheme for first-order derivatives requires for the time step to be of order $1/N^\alpha$ where N is the number of degrees of freedom in the r-direction and where $\alpha \approx 1.2$ for $N \leqslant 65$ and $\alpha \approx 2$ for $N \geqslant 65$. Consequently, it is not necessary to treat the first-order derivatives in an implicit (or semi-implicit) way when the number of points is less than 100. On the contrary, the high number of degrees of freedom allowed in the 2D case imposes that one treat the first-order operators in a semi-implicit way. We have then developed a semi-implicit finite difference temporal scheme for the 2D case and an explicit one, except for the diffusive terms which are treated implicitly, for the 3D case.

Application of these methods to the study of gravitational radiation emission during a 3D supernova collapse in the post-newtonian approximation by using a formalism developed by Blanchet *et al.* [13] is in progress. Another application will be the study of post-newtonian collapse with magnetic fields in order to model magnetic field in pulsars and neutron stars and possible bipolar jets.

The codes presented in this paper have been developed and tested on the VAX-8600 of the Observatoire de Paris, section de Meudon, as the 2.5D runs. Some of the 3D runs have been executed on the VP-200 of the Centre Interrégional de Calcul Electronique (Orsay).

APPENDIX A

1. *Associated Legendre Functions Expansion and Inversion of the Laplacian Operator in 3D*

Our method is based essentially on the expansion of physical quantities in Chebychev polynomials for the radial and the azimuthal parts. However, to invert the Laplacian operator in the 3D case, we used an expansion in spherical harmonics.

In order to get the Legendre coefficients, we first compute the expansion in Chebychev polynomials of first or second kind according to the parity of the longitudinal number m. The associated Legendre function expansion coefficients are then computed from the Chevychev expansion coefficients by a matrix multiplication.

Let $\mathbf{A}^m$ be the transformation matrix between associated Legendre functions P_l^m and Chebychev polynomials T_l expansion for a given m. This matrix is computed by using a Chebychev expansion of the associated Legendre function,

$$P_l^m(x) = \sum_{n=0}^{N} A_{nl}^m T_n(x), \tag{A.1}$$

where $x = \cos\theta$, $T_n(x) = \cos n\theta$ for even values of m and $x = \sin\theta$. $T_n(x) = \sin n\theta$ for odd values of m. The inverse matrix $\mathbf{B}^m$ is obtained by computing

$$B_{nl}^m = \int_{-1}^{+1} T_n(x)\, P_l^m(x)\, dx. \tag{A.2}$$

This integral is evaluated in Chebychev coefficients space by using recurrence relations. Note that this technique is much more efficient (in computing time as in accuracy) than a technique consisting of inverting the matrix $\mathbf{A}^m$.

The quantities being developed in spherical harmonics

$$Q(r, \theta, \varphi) = \sum_{l,m} Q_{lm}(r)\, Y_l^m(\theta, \varphi), \tag{A.3}$$

the Laplacian operator applied to the coefficients $Q_{lm}(r)$ reads

$$(\Delta Q)_{lm} = \Delta_l Q_{lm},$$

where

$$\Delta_l = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}. \tag{A.4}$$

Our problem consists of solving the system

$$Q_{lm}(r) - dt\, \Delta_l Q_{lm}(r) = \text{Source}_{lm}(r). \tag{A.5}$$

If the quantities $Q_{lm}(r)$ and $\text{Source}_{lm}(r)$ are expanded in Chebychev polynomials, the previous problem consist of solving an algebraic linear system of equations for the expansion coefficients Q_{ilm} and Source_{ilm} which are defined by

$$\begin{aligned} Q_{lm}(r) &= \sum_i Q_{ilm} T_i(r), \\ \text{Source}_{lm} &= \sum_i \text{Source}_{ilm} T_i(r), \end{aligned} \tag{A.6}$$

where $i = 2p$ or $i = 2p + 1$ accordingly to the parity of l. We are thus led in the construction and in the inversion of two kinds of matrices.

1.1. *l Even*

We have to distinguish two different cases $l = 0$ and $l \neq 0$. In the case $l = 0$, Δ_0 reads

$$\Delta_0 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}. \tag{A.7}$$

Consequently, $\Delta_0(T_i(r))$, where $T_i(r)$ is an even Chebychev polynomial (i even), is well defined. The matrix of the operator Δ_0 is obtained by

$$(\Delta_0)_{ij} = \langle T_i, \Delta_0(T_j) \rangle, \tag{A.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product associated to Chebychev polynomials.

However, for $l \neq 0$, it can be seen from (A.4) that Δ_l applied to a Chebychev polynomial T_i diverges at $r = 0$ and, consequently, that the matrix (A.8) is not defined. This is due to the fact that, as it has been explained in Section 2, the coefficients Q_{lm} have to vanish at $r = 0$ for $l > 0$. Taking into account this regularity property, according to the Galerkin approximation, we develop the quantities on a new set of independent polynomials $P_i(r)$ which vanishes at $r = 0$. We have chosen the set $P_i(r)$ to be

$$P_i(r) = T_{i+2}(r) - T_i(r). \tag{A.9}$$

Such a new basis will be call hereinafter a *Galerkin Basis*.

The matrix of the operator Δ_l is computed as

$$(\Delta_l)_{ij} = \langle T_i, \Delta_l(P_j)\rangle. \tag{A.10}$$

The system to be solved then reads

$$\sum_j (I_{ij} - dt(\Delta_l)_{ij}) Q_{jlm} = \text{Source}_{ilm}, \tag{A.11}$$

where I_{ij} is the transformation matrix between the Chebychev and Galerkin basis

1.2. *l Odd*

Two different cases $l = 1$ and $l \neq 1$ appear for l odd too. Recalling, that for i odd, $T_i(r)$ vanishes at $r = 0$, it can be seen that the operator Δ_l applied to T_i gives a finite value for $l = 1$. Consequently, the matrix

$$(\Delta_1)_{ij} = \langle T_i, \Delta_1(P_j)\rangle \tag{A.12}$$

is well defined.

However, for $l \geqslant 3$, $(\Delta_1)_{ij} = \langle T_i, \Delta_1(P_j)\rangle$ is not defined. So, in analogy to the *l-even* case, the quantities Q_{lm} have to be expanded on a *Galerkin Basis* of odd polynomials satisfying $P_i(0) = 0$ and $dP_i/dr\,|_{r=0} = 0$.

The boundary conditions are introduced by means of the τ-approximation. This approximation consist of replacing the last line of the matrix Δ_l by the line of coefficients (b_{lj}) determined in such a way that

$$\sum_{j=0}^{I} b_{Ij} Q_{jlm} = (\text{B.C.})_{lm} \tag{A.13}$$

is satisfied.

From a practical point of view, it should be noticed that the operator Δ_l can be reduce to a 7-band matrix by means of very simple algebraic manipulations. For that, we need a number of operations proportional to I.

2. *Inversion of the Laplacian in the 2D Case*

For the reasons explained in Section 3.1, the 2D quantities are expanded in Chebychev polynomials for the azimuthal part even in inverting the Laplacian. These polynomials being not eigenvalues of the Laplacian, a more tricky method for inverting this operator has to be developed.

Note first that two different operators appear, namely,

$$\Delta_0 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{A.14}$$

for the scalar-type quantities, which are expanded in first kind Chebychev polynomials ($\cos l\theta$) for the azimuthal part, and

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\right) \tag{A.15}$$

for vector-type quantities which are expanded in second kind Chebychev polynomials ($\sin l\theta$). Note that the above definitions differ from those of the 3D case. In what follows, we will describe how we invert Δ_0, the method used to invert Δ_1 being almost the same.

The problem is to solve the system

$$\left(\mathbf{1} + dt\left(\mathscr{R} + \frac{1}{r^2}\Theta\right)\right) Q(r,\theta) = \text{Source}(r,\theta), \tag{A.16}$$

where

$$\mathscr{R} = dt \times \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) \tag{A.17}$$

and

$$\Theta = dt \times \left(\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right). \tag{A.18}$$

After expansion of the quantity $Q(r,\theta)$ in Chebychev polynomials for θ, Eq. (A.16) becomes

$$\left[(\mathbf{1} + \mathscr{R})\delta_{lm} + \frac{\Theta_{ln}}{r^2}\right] Q_n(r) = \text{Source}_l(r), \tag{A.19}$$

or, in matricial notation,

$$(\delta_{lmnj} + dt\, \Delta_{lmnj}) Q_{lm} = \text{Source}_{nj}, \tag{A.20}$$

where Θ_{ln} is the matrix of the operator Θ in Chebychev representation.

The task of solving this system can be greatly facilitated by using the properties of Θ_{ln}. The linear combination

$$\Theta_{l,n} - \Theta_{l+2,n} \tag{A.21}$$

reduces the matrix Θ to a 2-band matrix, the only non-vanishing elements being $A_l = \Theta_{l,l}$ and $B_l = \Theta_{l,l+1}$.

The system (A.19) then reads

$$\left(1 + \mathscr{R} + \frac{A_l}{r^2}\right) Q_l(r) + \left(I_l + I_l\mathscr{R} + \frac{B_l}{r^2}\right) Q_{l+1}(r) = S_l(r), \tag{A.22}$$

where I_l and S_l are the elements of the representation of the identity matrix and the source terms after the linear combination (A.21).

The system of equations is solved by a descending recursion. The last equation reads

$$\left(1+\mathscr{R}+\frac{A_L}{r^2}\right)Q_L(r)=S_L(r), \tag{A.23}$$

where L is the maximum value of l. This equation has the same structure as Eq. (A.4) and can be solved by means of the same technique. Once Q_L is known, Q_{L-1} is computed and so on.

Note that the method described above can be generalised to more complicated operators such as, for instance, the 3D Laplacian or the operator $\Delta_1 - (\sin\theta/r)(\partial/\partial\theta)$ which appears in Eq. (51).

3. *Solution of the Poisson Equation*

Once the density has been expanded in spherical harmonics,

$$\rho(r,\theta,\varphi)=\sum \rho_{lm}(r)\,Y_l^m(\theta,\varphi),$$

the coefficients of the gravitational potential $\Phi_{lm}(r)$ must satisfy the Poisson equation

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)\Phi_{lm}(r)=-4\pi G\rho_{lm}(r) \tag{A.24}$$

and the boundary conditions $\Phi\to 0$ as $r\to\infty$. In our case, the boundary conditions have to be imposed at $r=1$. Consequently, we have to find an internal solution matching an external one which satisfies the boundary conditions at infinity. Note that, in our case, the external solution is a vacuum solution.

We first find a particular internal solution Φ_0 by inverting the Laplacian in Eq. (A.24) with the technique as described above. The general solution is $\Phi=\Phi_0+\Phi_h$, where we have assumed $\Phi_0=0$ for $r>1$ and where Φ_h is a harmonic function which has the coefficients

$$(\Phi_h)_{lm}=\alpha_l r^l; \qquad r\in[0,1],$$

and

$$(\Phi_h)_{lm}=\frac{\beta_l}{r^{l+1}}; \qquad r\in[1,\infty[. \tag{A.25}$$

We then determine the coefficients α_l and β_l in such a way that Φ is C^1-class.

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